

NORM ESTIMATES FOR THE BERGMAN AND CAUCHY-SZEGÖ PROJECTIONS OVER THE SIEGEL UPPER HALF-SPACE

CONGWEN LIU

ABSTRACT. We obtain several estimates for the L^p operator norms of the Bergman and Cauchy-Szegő projections over the the Siegel upper half-space. As a by-product, we also determine the precise value of the L^p operator norm of a family of integral operators over the Siegel upper half-space.

1. INTRODUCTION

Let \mathcal{U}^n be the Siegel upper half-space (or the generalized half-plane, following the terminology of Korányi [11, 12, 13, 14])

$$\mathcal{U}^n := \{z \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > |z'|^2\}$$

and let $\partial\mathcal{U}^n$ be its boundary in \mathbb{C}^{n+1} . Here and throughout, we use the notation

$$z = (z', z_{n+1}), \quad \text{where } z' = (z_1, \dots, z_n) \in \mathbb{C}^n \text{ and } z_{n+1} \in \mathbb{C}^1.$$

Note that $\mathcal{U}^0 = \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, the classical upper half-plane. \mathcal{U}^n is bi-holomorphically equivalent to the unit ball \mathbb{B}_{n+1} in \mathbb{C}^{n+1} , via the Cayley transform $\Phi : \mathbb{B}_{n+1} \rightarrow \mathcal{U}^n$ given by

$$(z', z_{n+1}) \mapsto \left(\frac{z'}{1 + z_{n+1}}, i \frac{1 - z_{n+1}}{1 + z_{n+1}} \right),$$

and so it is also referred to as the unbounded realization of the unit ball in \mathbb{C}^{n+1} .

As usual, for $p > 0$, the space $L^p(\mathcal{U}^n)$ consists of all Lebesgue measurable functions f on \mathcal{U}^n for which

$$\|f\|_p := \left\{ \int_{\mathcal{U}^n} |f(z)|^p dV(z) \right\}^{1/p}$$

is finite, where $dV = dm_{2n+2}$ is the Lebesgue measure on \mathbb{C}^{n+1} . The Bergman space $A^p(\mathcal{U}^n)$ is the closed subspace of $L^p(\mathcal{U}^n)$ consisting of holomorphic functions on \mathcal{U}^n . The orthogonal projection from $L^2(\mathcal{U}^n)$ onto $A^2(\mathcal{U}^n)$, known as the Bergman projection, can be expressed as an integral operator:

$$(P_{\mathcal{U}^n} f)(z) = \int_{\mathcal{U}^n} K_{\mathcal{U}^n}(z, w) f(w) dV(w),$$

2010 *Mathematics Subject Classification.* Primary 32A35, 47G10; Secondary 32A26, 30E20.

Key words and phrases. Siegel upper half-space; Bergman projection; Cauchy-Szegő projection; norm estimates.

This work was supported by the National Natural Science Foundation of China grants 11571333, 11471301.

with the Bergman kernel

$$(1.1) \quad K_{\mathcal{U}^n}(z, w) = \frac{(n+1)!}{4\pi^{n+1}} \left[\frac{i}{2}(\bar{w}_{n+1} - z_{n+1}) - z' \cdot \overline{w'} \right]^{-n-2}.$$

See [7, p.56, Lemma 5.1]. In the sequel, we shall use the notation

$$\rho(z, w) := \frac{i}{2}(\bar{w}_{n+1} - z_{n+1}) - z' \cdot \overline{w'}.$$

It has been long known that the Bergman projection $P_{\mathcal{U}^n}$ extends to a bounded operator from $L^p(\mathcal{U}^n)$ to $A^p(\mathcal{U}^n)$, for $1 < p < \infty$. See, for instance, [2, Lemma 2.8].

In this paper, we are concerned with estimates of the operator norm of $P_{\mathcal{U}^n}$ on $L^p(\mathcal{U}^n)$. Our first main result is the following.

Theorem 1. *For $1 < p < \infty$, we have*

$$(1.2) \quad \frac{\Gamma\left(\frac{n+2}{p}\right) \Gamma\left(\frac{n+2}{q}\right)}{\Gamma^2\left(\frac{n+2}{2}\right)} \leq \|P_{\mathcal{U}^n}\|_{p \rightarrow p} \leq \frac{(n+1)!}{\Gamma^2\left(\frac{n+2}{2}\right)} \frac{\pi}{\sin \frac{\pi}{p}},$$

where $q := \frac{p}{p-1}$ is the conjugate exponent of p and

$$\|P_{\mathcal{U}^n}\|_{p \rightarrow p} := \|P_{\mathcal{U}^n}\|_{L^p(\mathcal{U}^n) \rightarrow A^p(\mathcal{U}^n)} = \sup \left\{ \frac{\|P_{\mathcal{U}^n} f\|_p}{\|f\|_p} : f \in L^p(\mathcal{U}^n), f \neq 0 \right\}.$$

This is motivated by recent work of Zhu [30], Dostanić [4] and the author of the present paper [18], in which sharp estimates for the norm of the Bergman projection over the unit ball of \mathbb{C}^n were obtained. It is also worth mentioning that, in the recent years, there has been increasing interest in the study of the size of Bergman projection in various context other than the Bergman space. See [9, 10, 22, 23, 24, 28].

In the course of proving Theorem 1, we will precisely evaluate the L^p operator norm of a family of integral operators as follows. For $\alpha > -1$, we define

$$(T_\alpha f)(z) := \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \frac{\rho(w, w)^\alpha}{|\rho(z, w)|^{n+2+\alpha}} f(w) dV(w),$$

for, say, continuous f of compact support. It is a bounded map of $L^p(\mathcal{U}^n)$ to itself, as [2, Lemma 2.8] shown. Our second main result is the following.

Theorem 2. *If $1 \leq p < \infty$ and $p(1+\alpha) > 1$ then*

$$(1.3) \quad \|T_\alpha\|_{p \rightarrow p} = \frac{(n+1)! \Gamma\left(1+\alpha - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}.$$

Note that $\|P_{\mathcal{U}^n}\|_{p \rightarrow p} \leq \|T_0\|_{p \rightarrow p}$, and hence the second inequality in (1.2) follows immediately from (1.3), together with the well-known formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

We also remark that when $n = 0$, Theorem 2 gives an affirmative answer to a conjecture of Dostanić in [5] (a partial answer to this conjecture was given in [21]).

Recall that the Berezin transform over \mathcal{U}^n is defined by

$$\begin{aligned} (B_{\mathcal{U}^n} f)(z) &:= \int_{\mathcal{U}^n} \frac{|K_{\mathcal{U}^n}(z, w)|^2}{K_{\mathcal{U}^n}(z, z)} f(w) dV(w) \\ &= \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \frac{\rho(z, z)^{n+2}}{|\rho(z, w)|^{2(n+2)}} f(w) dV(w), \end{aligned}$$

which plays important roles in Berezin's theory of quantization as well as in the theory of Toeplitz operators. Note that $B_{\mathcal{U}^n} = T_{n+2}^*$, the adjoint of T_{n+2} . Another immediate consequence of Theorem 2 is the following.

Corollary 3. *If $1 < p \leq \infty$ then*

$$\|B_{\mathcal{U}^n}\|_{p \rightarrow p} = \frac{\pi}{p \sin \frac{\pi}{p}} \prod_{k=1}^{n+1} \left(1 + \frac{1}{kp}\right).$$

When $p = \infty$, the quantity on the right hand side should be interpreted as 1.

This is an analogue of the main results in [3] and [20].

Our third main result concerns the Cauchy-Szegő projection over $b\mathcal{U}^n$. For f holomorphic on \mathcal{U}^n , we define

$$\|f\|_{H^p} := \sup_{t>0} \left\{ \int_{b\mathcal{U}^n} |f(u + t\mathbf{i})|^p d\beta(u) \right\}^{\frac{1}{p}},$$

where $\mathbf{i} = (0', i) \in \mathbb{C}^{n+1}$ and the measure $d\beta$ on $b\mathcal{U}^n$ is defined by the formula

$$\int_{b\mathcal{U}^n} f d\beta = \int_{\mathbb{C}^n \times \mathbb{R}} f(z', t + i|z'|^2) dz' dt,$$

for (say) continuous f of compact support. See Section 2.1 below. Then we set

$$H^p(\mathcal{U}^n) := \{f : f \text{ is holomorphic on } \mathcal{U}^n, \|f\|_{H^p} < \infty\},$$

which is the analogue for \mathcal{U}^n of the classical Hardy space H^p of holomorphic functions in the upper half-plane.

The space $H^2(\mathcal{U}^n)$ can be identified with the closed subspace of $L^2(b\mathcal{U}^n)$ consisting of functions $\{f^b\}$ that are boundary values of functions $f \in H^2(\mathcal{U}^n)$, so there exists an orthogonal projection from $L^2(b\mathcal{U}^n)$ onto $H^2(\mathcal{U}^n)$. We denote this projection by $C_{\mathcal{U}^n}$ and call it the Cauchy-Szegő projection. It may also be written as the Cauchy-Szegő integral

$$(C_{\mathcal{U}^n} f)(z) = \int_{b\mathcal{U}^n} S_{\mathcal{U}^n}(z, u) f(u) d\beta(u), \quad z \in \mathcal{U}^n,$$

where

$$(1.4) \quad S_{\mathcal{U}^n}(z, w) = \frac{n!}{4\pi^{n+1}} \rho(z, w)^{-n-1}.$$

See [7, p.61, Proposition 5.3] or [27, p.536, Theorem 1]. A classical theorem by Korányi and Vági [13, Theorem 6.1] asserts that the Cauchy-Szegő projection has an extension to a bounded operator from $L^p(b\mathcal{U}^n)$ onto $H^p(\mathcal{U}^n)$, for $1 < p < \infty$.

Our third main result gives a lower bound for the operator norm of $C_{\mathcal{U}^n}$.

Theorem 4. For all $1 < p < \infty$,

$$(1.5) \quad \|C_{\mathcal{U}^n}\|_{L^p(\mathcal{BU}^n) \rightarrow H^p(\mathcal{U}^n)} \geq \frac{\Gamma\left(\frac{n+1}{p}\right) \Gamma\left(\frac{n+1}{q}\right)}{\Gamma^2\left(\frac{n+1}{2}\right)},$$

where $q := \frac{p}{p-1}$ is the conjugate exponent of p .

We shall deduce Theorem 4 from Theorem 1, with the help of the following inequality, which makes a connection between the norms of the two operators and might be of independent interest.

Theorem 5. For all $1 < p < \infty$,

$$(1.6) \quad \|C_{\mathcal{U}^n}\|_{L^p(\mathcal{BU}^n) \rightarrow H^p(\mathcal{U}^n)} \geq \|P_{\mathcal{U}^{n-1}}\|_{L^p(\mathcal{U}^{n-1}) \rightarrow A^p(\mathcal{U}^{n-1})}.$$

The above results suggest the following.

Conjecture 6. For all $1 < p < \infty$, we have

$$(1.7) \quad \|P_{\mathcal{U}^n}\|_{L^p(\mathcal{U}^n) \rightarrow A^p(\mathcal{U}^n)} = \frac{\Gamma\left(\frac{n+2}{p}\right) \Gamma\left(\frac{n+2}{q}\right)}{\Gamma^2\left(\frac{n+2}{2}\right)}$$

and

$$(1.8) \quad \|C_{\mathcal{U}^n}\|_{L^p(\mathcal{BU}^n) \rightarrow H^p(\mathcal{U}^n)} = \frac{\Gamma\left(\frac{n+1}{p}\right) \Gamma\left(\frac{n+1}{q}\right)}{\Gamma^2\left(\frac{n+1}{2}\right)},$$

where $q := \frac{p}{p-1}$ is the conjugate exponent of p .

Note that when $n = 0$, the conjectured (1.8) reads

$$(1.9) \quad \|C_{\mathbb{C}^+}\|_{L^p(\mathbb{R}) \rightarrow H^p(\mathbb{C}^+)} = \frac{1}{\sin \frac{\pi}{p}}.$$

This coincides with a variant of the Gohberg-Krupnik conjecture, which was proved by Hollenbeck and Verbitsky [8] only in 2000. For the proof of (1.9), see [17, p.373]. This provides a support for our conjecture. It is also noteworthy that the conjectured (1.8) would imply (1.7), in view of Theorems 1 and 5. See also [18] and [19] for their counterparts in the setting of the unit ball.

The rest of the paper is organized as follows: In Section 2 we review some definitions and basic facts, and in Section 3 we establish several technical lemmas, some of them might be of independent interest. Sections 4 is devoted to the proof of Theorem 2. Our first main result, Theorem 1 will be proved in Sections 5. Sections 6 is devoted to the proof of Theorem 5.

Acknowledgements. The author is indebted to Lifang Zhou for correcting two errors in an earlier version of the paper, and to Guangbin Ren and Xieping Wang for many helpful comments.

2. PRELIMINARIES

2.1. Heisenberg group. We recall the definition of the Heisenberg group and some basic facts which can be found in [27, Chapter XII] and [16, Chapter 9].

We denote by \mathbb{H}^n the Heisenberg group, that is, the set

$$\mathbb{C}^n \times \mathbb{R} = \{[\zeta, t] : \zeta \in \mathbb{C}^n, t \in \mathbb{R}\}$$

endowed with the Heisenberg group operation

$$[\zeta, t] \cdot [\eta, s] = [\zeta + \eta, t + s + 2\operatorname{Im}(\zeta \cdot \bar{\eta})].$$

Here we shall use square brackets $[\]$ for elements of the Heisenberg group to distinguish them from points in \mathbb{C}^{n+1} , for which parentheses $(\)$ are used. The identity element is $[0, 0]$ and the inverse of $[\zeta, t]$ is $[\zeta, t]^{-1} = [-\zeta, -t]$. The Haar measure dh on \mathbb{H}^n is the usual Lebesgue measure $d\zeta dt$ on $\mathbb{C}^n \times \mathbb{R}$, here we write $h = [\zeta, t]$, $\zeta \in \mathbb{C}^n, t \in \mathbb{R}$. (To be more precise, $d\zeta = d\eta d\xi$ if $\zeta = \eta + i\xi$ with $\eta, \xi \in \mathbb{R}^n$.)

To each element $h = [\zeta, t]$ of \mathbb{H}^n , we associate the following (holomorphic) affine self-mapping of \mathcal{U}^n :

$$(2.1) \quad h : (z', z_{n+1}) \mapsto (z' + \zeta, z_{n+1} + t + 2iz' \cdot \bar{\zeta} + i|\zeta|^2).$$

The mappings (2.1) are simply transitive on the boundary $b\mathcal{U}^n$ of \mathcal{U}^n , so we can identify the Heisenberg group with $b\mathcal{U}^n$ via its action on the origin

$$\mathbb{H}^n \ni [\zeta, t] \mapsto (\zeta, t + i|\zeta|^2) \in b\mathcal{U}^n.$$

This identification allows us to transport the Haar measure dh on \mathbb{H}^n to a measure $d\beta$ on $b\mathcal{U}^n$; that is, we have the integration formula

$$(2.2) \quad \int_{b\mathcal{U}^n} f d\beta = \int_{\mathbb{C}^n \times \mathbb{R}} f(z', t + i|z'|^2) dz' dt,$$

for (say) continuous f of compact support. The measure $d\beta$ is invariant under the action of \mathbb{H}^n , that is, $d\beta(h(z)) = d\beta(z)$ for each $h \in \mathbb{H}^n$.

We shall make frequent use of the following Fubini's theorem on \mathcal{U}^n :

$$(2.3) \quad \int_{\mathcal{U}^n} f dV = \int_0^\infty \int_{b\mathcal{U}^n} f(u + ti) d\beta(u) dt,$$

which is valid for, say, continuous f of compact support. This can be easily verified by substituting (2.2) into the right hand side of (2.3)

Finally, it is easy to verify that

$$(2.4) \quad \rho(h(z), h(w)) = \rho(z, w),$$

for each $h \in \mathbb{H}^n$, and for all $z \in \mathcal{U}^n, w \in b\mathcal{U}^n$.

2.2. Cayley transform. Recall that the Cayley transform $\Phi : \mathbb{B}_{n+1} \rightarrow \mathcal{U}^n$ is given by

$$(z', z_{n+1}) \mapsto \left(\frac{z'}{1 + z_{n+1}}, i \left(\frac{1 - z_{n+1}}{1 + z_{n+1}} \right) \right).$$

It is immediate to calculate that

$$\Psi := \Phi^{-1} : (z', z_{n+1}) \mapsto \left(\frac{2iz'}{i + z_{n+1}}, \frac{i - z_{n+1}}{i + z_{n+1}} \right).$$

Again, we refer to [27, Chapter XII] and [16, Chapter 9] for the properties of these two mappings. For the convenience of later reference, we record the following lemma.

Lemma 7. *The mappings Φ and Ψ have the following elementary properties:*

(i) *The identity*

$$(2.5) \quad \rho(\Phi(\xi), \Phi(\eta)) = \frac{1 - \xi \cdot \bar{\eta}}{(1 + \xi_{n+1})(1 + \bar{\eta}_{n+1})}$$

holds for all $\xi, \eta \in \mathbb{B}_{n+1}$.

(ii) *The real Jacobian of Φ at $\xi \in \mathbb{B}_{n+1}$ is*

$$(2.6) \quad (J_R \Phi)(\xi) = \frac{4}{|1 + \xi_{n+1}|^{2(n+2)}}.$$

(iii) *The identity*

$$(2.7) \quad 1 - \Psi(z) \cdot \overline{\Psi(w)} = \frac{\rho(z, w)}{\rho(z, \mathbf{i})\rho(\mathbf{i}, w)}$$

holds for all $z, w \in \mathcal{U}^n$, where $\mathbf{i} = (0', i)$.

(iv) *The identity*

$$(2.8) \quad |\Psi(z)|^2 = 1 - \frac{\rho(z, z)}{|\rho(z, \mathbf{i})|^2}$$

holds for all $z \in \mathcal{U}^n$.

(v) *The real Jacobian of Ψ at $z \in \mathcal{U}^n$ is*

$$(2.9) \quad (J_R \Psi)(z) = \frac{1}{4|\rho(z, \mathbf{i})|^{2(n+2)}}.$$

Note that the mappings Φ and Ψ extend also to the boundaries of the domains \mathbb{B}_{n+1} and \mathcal{U}^n . Thus, \mathcal{BU}^n corresponds via Ψ to the unit sphere \mathbb{S}_{n+1} , except for the “south pole” $(0', -1)$. It is easy to check that the identity

$$(2.10) \quad \rho(z, w) = \frac{1 - \Psi(z) \cdot \overline{\Psi(w)}}{(1 + [\Psi(z)]_{n+1})(1 + [\overline{\Psi(w)}]_{n+1})}$$

holds for all $z \in \mathcal{U}^n$ and $w \in \mathcal{BU}^n$.

Finally, writing $d\sigma$ for the normalized surface measure on the unit sphere \mathbb{S}_{n+1} , one has the following change of variables formula (see [27, p.575, 7.2(b)]):

$$(2.11) \quad \int_{\mathcal{BU}^n} f d\beta = \frac{4\pi^{n+1}}{n!} \int_{\mathbb{S}_{n+1}} \frac{f(\Phi(\xi))}{|1 + \xi_{n+1}|^{2(n+1)}} d\sigma(\xi).$$

2.3. Möbius transformations. The group of all one-to-one holomorphic mappings of \mathbb{B}_{n+1} onto \mathbb{B}_{n+1} (the so-called automorphisms of \mathbb{B}_{n+1}) will be denoted by $\text{Aut}(\mathbb{B}_{n+1})$. It is generated by the unitary transformations on \mathbb{C}^{n+1} along with the Möbius transformations φ_ξ given by

$$\varphi_\xi(\eta) := \frac{\xi - P_\xi \eta - (1 - |\xi|^2)^{\frac{1}{2}} Q_\xi \eta}{1 - \eta \cdot \bar{\xi}},$$

where $\xi \in \mathbb{B}_{n+1}$, P_ξ is the orthogonal projection onto the space spanned by ξ , and $Q_\xi \eta = \eta - P_\xi \eta$.

It is easily shown that the mapping φ_ξ satisfies

$$\varphi_\xi(0) = \xi, \quad \varphi_\xi(\xi) = 0, \quad \varphi_\xi(\varphi_\xi(\eta)) = \eta.$$

Furthermore, for all $\eta, \omega \in \overline{\mathbb{B}_{n+1}}$,

$$(2.12) \quad 1 - \varphi_\xi(\eta) \cdot \overline{\varphi_\xi(\omega)} = \frac{(1 - |\xi|^2)(1 - \eta \cdot \overline{\omega})}{(1 - \eta \cdot \overline{\xi})(1 - \xi \cdot \overline{\omega})},$$

and in particular,

$$(2.13) \quad 1 - \varphi_\xi(\eta) \cdot \overline{\xi} = \frac{1 - |\xi|^2}{1 - \eta \cdot \overline{\xi}}.$$

Finally, an easy computation shows that

$$(2.14) \quad 1 - \varphi_\xi(\eta) \cdot \overline{\omega} = \frac{(1 - \eta \cdot \overline{\varphi_\xi(\omega)})(1 - \xi \cdot \overline{\omega})}{1 - \eta \cdot \overline{\xi}}$$

holds for all $\eta, \omega \in \overline{\mathbb{B}_{n+1}}$.

The best general reference here is [26, Chapter 2].

2.4. Hypergeometric functions. We use the classical notation

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda \right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\lambda^k}{k!}$$

with $c \neq 0, -1, -2, \dots$, where

$$(a)_0 = 1, \quad (a)_k = a(a+1) \dots (a+k-1) \quad \text{for } k \geq 1.$$

denotes the Pochhammer symbol of a . This series gives an analytic function for $|\lambda| < 1$, called the Gauss hypergeometric function associated to (a, b, c) .

We refer to [1, Chapter 2] for the properties of these functions. Here, we only record three formulas for later reference.

$$(2.15) \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0.$$

$$(2.16) \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda \right] = (1-\lambda)^{c-a-b} {}_2F_1 \left[\begin{matrix} c-a, c-b \\ c \end{matrix} ; \lambda \right].$$

$$(2.17) \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda \right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t\lambda)^{-a} dt, \quad \text{Re } c > \text{Re } b > 0; |\arg(1-\lambda)| < \pi.$$

2.5. Schur's test. The following lemma, usually called Schur's test, is one of the most commonly used results for proving the L^p -boundedness of integral operators. See, for example, [31].

Lemma 8. *Suppose that (X, μ) is a σ -finite measure space and $Q(x, y)$ is a non-negative measurable function on $X \times X$ and T is the associated integral operator*

$$Tf(x) = \int_X Q(x, y) f(y) d\mu(y).$$

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If there exist a positive constant C and a positive measurable function g on X such that

$$\int_X Q(x, y) g(y)^q d\mu(y) \leq C g(x)^q$$

for almost every x in X and

$$\int_X Q(x, y) g(x)^p d\mu(x) \leq C g(y)^p$$

for almost every y in X , then T is bounded on $L^p(X, \mu)$ with $\|T\| \leq C$.

3. TECHNICAL LEMMAS

3.1. An elementary inequality.

Lemma 9. *Let $a, b \in \mathbb{R}$ and $c > \max\{a + b + 1, a + 1, b\}$. Then*

$$(3.1) \quad \left| {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda \right] - {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; 1 \right] \right| \leq C(a, b, c) \cdot |1 - \lambda|$$

for all $\lambda \in \mathbb{D}$, where

$$C(a, b, c) := |a| \cdot \max \left\{ 2^{-a-1}, \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a-1)\Gamma(c-b)} \right\}.$$

Proof. Since $c > b$ and $c - a - b > 0$, by (2.17), we have

$$(3.2) \quad \begin{aligned} & \left| {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda \right] - {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; 1 \right] \right| \\ & \leq \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} |(1-t\lambda)^{-a} - (1-t)^{-a}| dt \end{aligned}$$

for all $\lambda \in \mathbb{D}$. By the mean value theorem we have

$$\begin{aligned} |(1-t\lambda)^{-a} - (1-t)^{-a}| & \leq \sup_{\vartheta \in t\mathbb{D}} |a(1-\vartheta)^{-a-1}| \cdot |1-\lambda| \\ & \leq |a| \cdot \max \{ 2^{-a-1}, (1-t)^{-a-1} \} |1-\lambda|. \end{aligned}$$

Substituting this into (3.2) yields the desired inequality (3.1). \square

3.2. A Forelli-type formula. The following lemma deals with integration on $b\mathcal{U}^n$ of functions of fewer variables, which is in the same spirit as a result of Forelli [6, p.383] (see also [26, p.14] or [29, p.10, Lemma 1.9]).

Suppose $0 \leq k \leq n$ and let \mathcal{U}^k be the Siegel upper half-space in \mathbb{C}^{k+1} . It is convenient to let

$$\rho_k(w) := \operatorname{Im} w_{k+1} - \sum_{j=1}^k |w_j|^2$$

We think of ρ_k as a “height function” in \mathcal{U}^k . Note that

$$\rho_n(z) = \rho(z, z) = \operatorname{Im} z_{n+1} - |z'|^2.$$

Lemma 10. *Suppose $0 \leq k < n$ and f is a function on $b\mathcal{U}^n$ that depends only on $z_{n-k+1}, \dots, z_{n+1}$. Then f can be regarded as defined on \mathcal{U}^k and*

$$\int_{b\mathcal{U}^n} (f \circ \Pi_k) d\beta = \frac{\pi^{n-k}}{(n-k-1)!} \int_{\mathcal{U}^k} \rho_k(w)^{n-k-1} f(w) dm_{2k+2}(w),$$

where Π_k is the orthogonal projection of \mathbb{C}^{n+1} onto \mathbb{C}^{k+1} given by

$$(z_1, \dots, z_{n+1}) \longmapsto (z_{n-k+1}, \dots, z_{n+1}),$$

and

$$c_{n,k} := \frac{\pi^{n-k}}{(n-k-1)!}.$$

In particular when $k = n - 1$, this reads

$$(3.3) \quad \int_{b\mathcal{U}^n} (f \circ \Pi_{n-1}) d\beta = \pi \int_{\mathcal{U}^{n-1}} f(w) dm_{2n}(w).$$

Proof. For convenience, we use the notation $z = (z^\dagger, z^\ddagger)$, where

$$z^\dagger = (z_1, \dots, z_{n-k}) \in \mathbb{C}^{n-k} \quad \text{and} \quad z^\ddagger = (z_{n-k+1}, \dots, z_{n+1}) \in \mathbb{C}^{k+1}.$$

By an approximation argument, it suffices for us to prove the result when f is continuous in \mathbb{C}^{k+1} and has support in $\{z^\ddagger \in \mathbb{C}^{k+1} : \rho_k(z^\ddagger) > r_0\}$ for some $r_0 > 0$.

Fix such an f and consider the integrals

$$I(r) = \int_{\{\rho_n(z) > r\}} (f \circ \Pi_k) dV, \quad 0 < r < \infty.$$

By Fubini's theorem (2.3), we have

$$I(r) = \int_r^\infty \left\{ \int_{b\mathcal{U}^n} (f \circ \Pi_k)(u + t\mathbf{i}) d\beta(u) \right\} dt.$$

We then differentiate this to obtain

$$(3.4) \quad I'(0) = - \int_{b\mathcal{U}^n} (f \circ \Pi_k)(u) d\beta(u).$$

On the other hand, if $0 < r < r_0$, an application of the classical Fubini's theorem shows that

$$\begin{aligned} I(r) &= \int_{\{\rho_k(z^\ddagger) > r\}} \left\{ \int_{\{|z^\dagger|^2 < \rho_k(z^\ddagger) - r\}} (f \circ \Pi_k)(z^\dagger, z^\ddagger) dm_{2n-2k}(z^\dagger) \right\} dm_{2k+2}(z^\ddagger) \\ &= \int_{\{\rho_k(z^\ddagger) > r\}} \frac{\pi^{n-k}}{(n-k)!} (\rho_k(z^\ddagger) - r)^{n-k} f(z^\ddagger) dm_{2k+2}(z^\ddagger) \\ &= \frac{\pi^{n-k}}{(n-k)!} \int_{\mathcal{U}^k} (\rho_k(z^\ddagger) - r)^{n-k} f(z^\ddagger) dm_{2k+2}(z^\ddagger), \end{aligned}$$

where the last equality follows from the assumption that f is supported in $\{z^\ddagger \in \mathbb{C}^{k+1} : \rho_k(z^\ddagger) > r_0\}$. Differentiation then gives

$$(3.5) \quad I'(0) = - \frac{\pi^{n-k}}{(n-k-1)!} \int_{\mathcal{U}^k} \rho_k(z^\ddagger)^{n-k-1} f(z^\ddagger) dm_{2k+2}(z^\ddagger).$$

Comparison of (3.4) and (3.5) gives (10). \square

Corollary 11. *Let f be a function of one complex variable. Then, for any $z \in b\mathcal{U}^n$, we have*

$$(3.6) \quad \int_{b\mathcal{U}^n} f(2i\rho(w, z)) d\beta(w) = \frac{\pi^n}{(n-1)!} \int_{\mathbb{C}_+} f(\lambda) (\operatorname{Im} \lambda)^{n-1} dm_2(\lambda).$$

Proof. Let $z \in b\mathcal{U}^n$ be fixed. We put $h := [-z', -z_{n+1} + i|z'|^2] \in \mathbb{H}^n$. Then $h(z) = 0$. Hence, by (2.4), we see that

$$\begin{aligned} \int_{b\mathcal{U}^n} f(2i\rho(w, z))d\beta(w) &= \int_{b\mathcal{U}^n} f(2i\rho(h(w), 0))d\beta(w) \\ &= \int_{b\mathcal{U}^n} f(2i\rho(u, 0))d\beta(u) = \int_{b\mathcal{U}^n} f(u_{n+1})d\beta(u), \end{aligned}$$

where the second equality follows from the \mathbb{H}^n -invariance of $d\beta$. Finally, an application of Lemma 10 (with $k = 0$) completes the proof. \square

Corollary 12. *Suppose $1 \leq k < n$. Then for $f \in L^1(b\mathcal{U}^n)$ we have*

$$\int_{b\mathcal{U}^n} f d\beta = c_{n,k} \int_{\mathcal{U}^k} \left\{ \int_{\mathbb{S}_{n-k}} f(\sqrt{\rho_k(w)} \eta, w) d\sigma_{n-k}(\eta) \right\} \rho_k(w)^{n-k-1} dm_{2k+2}(w).$$

In particular, when $k = n - 1$,

$$(3.7) \quad \int_{b\mathcal{U}^n} f d\beta = \pi \int_{\mathcal{U}^{n-1}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{\rho_{n-1}(w)} e^{i\theta}, w) d\theta \right\} dm_{2n}(w).$$

Proof. As in the proof of Lemma 10, we write $z = (z^\dagger, z^\ddagger)$, where $z^\dagger \in \mathbb{C}^{n-k}$ and $z^\ddagger \in \mathbb{C}^{k+1}$. Then

$$\int_{b\mathcal{U}^n} f d\beta = \int_{b\mathcal{U}^n} f(z^\dagger, z^\ddagger) d\beta(z).$$

Note that if $z = (z^\dagger, z^\ddagger) \in b\mathcal{U}^n$ then $|z^\dagger|^2 = \rho_k(z^\ddagger)$. Since $d\beta(\mathbf{g}(z)) = d\beta(z)$ for every unitary linear transformation \mathbf{g} on \mathbb{C}^n , where $\mathbf{g}((z', z_{n+1})) := (\mathbf{g}(z'), z_{n+1})$,

$$\int_{b\mathcal{U}^n} f(z^\dagger, z^\ddagger) d\beta(z) = \int_{b\mathcal{U}^n} f(|z^\dagger| \eta, z^\ddagger) d\beta(z) = \int_{b\mathcal{U}^n} f(\sqrt{\rho_k(z^\ddagger)} \eta, z^\ddagger) d\beta(z)$$

for any fixed $\eta \in \mathbb{S}_{n-k}$. Integrating over $\eta \in \mathbb{S}_{n-k}$ and applying Fubini's theorem, we obtain

$$\int_{b\mathcal{U}^n} f d\beta = \int_{b\mathcal{U}^n} \left\{ \int_{\mathbb{S}_{n-k}} f(\sqrt{\rho_k(z^\ddagger)} \eta, z^\ddagger) d\sigma_{n-k}(\eta) \right\} d\beta(z).$$

The inner integral above defines a function that only depends on the last $k + 1$ variables. Therefore, an application of Lemma 10 completes the proof. \square

3.3. Evaluation of some integrals.

Lemma 13. *Let $\theta > 0$ and $\gamma > -1$. The identities*

$$(3.8) \quad \int_{b\mathcal{U}^n} \frac{d\beta(u)}{|\rho(z, u)|^{n+1+\theta}} = \frac{4\pi^{n+1}\Gamma(\theta)}{\Gamma^2\left(\frac{n+1+\theta}{2}\right)} \rho(z, z)^{-\theta}$$

and

$$(3.9) \quad \int_{\mathcal{U}^n} \frac{\rho(w, w)^\gamma dV(w)}{|\rho(z, w)|^{n+2+\theta+\gamma}} = \frac{4\pi^{n+1}\Gamma(1+\gamma)\Gamma(\theta)}{\Gamma^2\left(\frac{n+2+\theta+\gamma}{2}\right)} \rho(z, z)^{-\theta}$$

hold for all $z \in \mathcal{U}^n$.

Proof. For fixed $z \in \mathcal{U}^n$, we put $h = [-z', -\operatorname{Re} z_{n+1}] \in \mathbb{H}^n$. Recall that $h \in \mathbb{H}^n$ acts on $z \in \mathcal{U}^n$ by (2.1). It is easy to check that $h(z) = (0, i\rho(z, z))$. Note also that an element of $b\mathcal{U}^n$ has the form $(w', t + i|w'|^2)$ with $t = \operatorname{Re} w_{n+1}$. It follows that

$$\rho(h(z), w) = \frac{|w'|^2 + \rho(z, z) + it}{2}$$

for every $w \in b\mathcal{U}^n$. Since $d\beta$ is \mathbb{H}^n -invariant, by making the change of variables $w \mapsto h^{-1}(w)$ and using (2.4), we obtain

$$\begin{aligned} \int_{b\mathcal{U}^n} \frac{d\beta(w)}{|\rho(z, w)|^{n+1+\theta}} &= \int_{b\mathcal{U}^n} \frac{d\beta(w)}{|\rho(h(z), w)|^{n+1+\theta}} \\ &= 2^{n+1+\theta} \int_{\mathbb{C}^n \times \mathbb{R}} \frac{dw' dt}{[|w'|^2 + \rho(z, z) + t^2]^{\frac{n+1+\theta}{2}}}. \end{aligned}$$

A simple scaling argument (which involves carrying out the t -integration first) shows that the last integral equals

$$\begin{aligned} &\left\{ \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\frac{n+1+\theta}{2}}} \right\} \left\{ \int_0^{\infty} \frac{r^{2n-1}}{(1+r^2)^{n+\theta}} dr \right\} \frac{2\pi^n}{\Gamma(n)} \rho(z, z)^{-\theta} \\ &= \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{n+\theta}{2})}{\Gamma(\frac{n+1+\theta}{2})} \frac{\Gamma(n)\Gamma(\theta)}{2\Gamma(n+\theta)} \frac{2\pi^n}{\Gamma(n)} \rho(z, z)^{-\theta} \\ &= 2^{1-n-\theta} \frac{\pi^{n+1}\Gamma(\theta)}{\Gamma^2(\frac{n+1+\theta}{2})} \rho(z, z)^{-\theta}. \end{aligned}$$

Here, in the first equality we have used the well-known identity

$$\int_0^{\infty} \frac{t^{2b-1}}{(1+t^2)^a} dt = \frac{\Gamma(b)\Gamma(a-b)}{2\Gamma(a)}, \quad \text{if } a > b,$$

and, in the second equality we have used the “duplication formula”

$$\Gamma\left(\frac{1}{2}\right) \Gamma(2a) = 2^{2a-1} \Gamma(a) \Gamma\left(a + \frac{1}{2}\right).$$

This proves (3.8).

We proceed to prove (3.9). By (2.3), we have

$$\int_{\mathcal{U}^n} \frac{\rho(u, u)^\gamma du}{|\rho(z, u)|^{n+2+\theta+\gamma}} = \int_0^{\infty} \left\{ \int_{b\mathcal{U}^n} \frac{d\beta(w)}{|\rho(z, w + t\mathbf{i})|^{n+2+\theta+\gamma}} \right\} t^\gamma dt.$$

Note that $\rho(z, w + t\mathbf{i}) = \rho(z + t\mathbf{i}, w)$ and $\rho(z + t\mathbf{i}, z + t\mathbf{i}) = \rho(z, z) + t$. Applying (3.8) to the inner integral yields

$$\begin{aligned} \int_{\mathcal{U}^n} \frac{\rho(u, u)^\gamma du}{|\rho(z, u)|^{n+2+\theta+\gamma}} &= \frac{4\pi^{n+1}\Gamma(1+\theta+\gamma)}{\Gamma^2\left(\frac{n+2+\theta+\gamma}{2}\right)} \int_0^{\infty} \frac{t^\gamma dt}{(\rho(z, z) + t)^{1+\theta+\gamma}} \\ &= \frac{4\pi^{n+1}\Gamma(1+\theta+\gamma)}{\Gamma^2\left(\frac{n+2+\theta+\gamma}{2}\right)} \frac{\Gamma(1+\gamma)\Gamma(\theta)}{\Gamma(1+\theta+\gamma)} \rho(z, z)^{-\theta}, \end{aligned}$$

as desired. \square

Lemma 14. (i) If $a \in \mathbb{R}$ and $\max\{b, c, b+c\} < n+1$, then

$$(3.10) \quad \begin{aligned} & \int_{\mathbb{S}_{n+1}} \frac{d\sigma(\omega)}{(1-\eta \cdot \bar{\omega})^a (1-\zeta \cdot \bar{\omega})^b (1-\omega \cdot \bar{\zeta})^c} \\ &= \frac{n! \Gamma(n+1-b-c)}{\Gamma(n+1-b) \Gamma(n+1-c)} {}_2F_1 \left[\begin{matrix} a, c \\ n+1-b \end{matrix} ; \eta \cdot \bar{\zeta} \right] \end{aligned}$$

holds for any $\eta \in \mathbb{B}_{n+1}$ and $\zeta \in \mathbb{S}_{n+1}$.

(ii) If $a \in \mathbb{R}$ and $\max\{b, c, b+c\} < n+2$, then

$$(3.11) \quad \begin{aligned} & \int_{\mathbb{B}_{n+1}} \frac{dV(\xi)}{(1-\eta \cdot \bar{\xi})^a (1-\zeta \cdot \bar{\xi})^b (1-\xi \cdot \bar{\zeta})^c} \\ &= \frac{\pi^{n+1} \Gamma(n+2-b-c)}{\Gamma(n+2-b) \Gamma(n+2-c)} {}_2F_1 \left[\begin{matrix} a, c \\ n+2-b \end{matrix} ; \eta \cdot \bar{\zeta} \right] \end{aligned}$$

holds for any $\eta \in \mathbb{B}_{n+1}$ and $\zeta \in \mathbb{S}_{n+1}$.

Proof. We may further assume that $b+c < 0$; if we prove the lemma in this special case, the general case follows by analytic continuation. We prove only the first part of the lemma, the proof of the second part being similar.

According to [18, Lemma 2.3], the identity

$$\begin{aligned} & \int_{\mathbb{S}_{n+1}} \frac{d\sigma(\omega)}{(1-\eta \cdot \bar{\omega})^a (1-r\zeta \cdot \bar{\omega})^b (1-r\omega \cdot \bar{\zeta})^c} \\ &= \sum_{j=0}^{\infty} \frac{(a)_j (c)_j}{(n+1)_j j!} {}_2F_1 \left[\begin{matrix} b, c+j \\ n+1+j \end{matrix} ; r^2 \right] (r\eta \cdot \bar{\zeta})^j \end{aligned}$$

holds for all $r \in [0, 1)$, $\eta \in \mathbb{B}_{n+1}$ and $\zeta \in \mathbb{S}_{n+1}$. Note that

$$\left| \frac{1}{(1-\eta \cdot \bar{\omega})^a (1-r\zeta \cdot \bar{\omega})^b (1-r\omega \cdot \bar{\zeta})^c} \right| \leq \frac{2^{-b-c}}{|1-\eta \cdot \bar{\omega}|^a},$$

since $b+c < 0$. Letting $r \rightarrow 1$, applying the dominated convergence theorem and using (2.15), we obtain

$$\begin{aligned} & \int_{\mathbb{S}_{n+1}} \frac{d\sigma(\omega)}{(1-\eta \cdot \bar{\omega})^a (1-\zeta \cdot \bar{\omega})^b (1-\omega \cdot \bar{\zeta})^c} \\ &= \sum_{j=0}^{\infty} \frac{(a)_j (c)_j}{(n+1)_j j!} {}_2F_1 \left[\begin{matrix} b, c+j \\ n+1+j \end{matrix} ; 1 \right] (\eta \cdot \bar{\zeta})^j \\ &= \sum_{j=0}^{\infty} \frac{(a)_j (c)_j}{(n+1)_j j!} \frac{\Gamma(n+1+j) \Gamma(n+1-b-c)}{\Gamma(n+1-b+j) \Gamma(n+1-c)} (\eta \cdot \bar{\zeta})^j \\ &= \frac{n! \Gamma(n+1-b-c)}{\Gamma(n+1-b) \Gamma(n+1-c)} {}_2F_1 \left[\begin{matrix} a, c \\ n+1-b \end{matrix} ; \eta \cdot \bar{\zeta} \right] \end{aligned}$$

as desired. \square

Lemma 15. For $\theta \in \mathbb{R}$ and $\gamma > -1$, the identity

$$(3.12) \quad \int_{\mathbb{B}_{n+1}} \frac{(1-|\xi|^2)^\gamma dV(\xi)}{|1-\eta \cdot \bar{\xi}|^{2\theta}} = \frac{\pi^{n+1} \Gamma(1+\gamma)}{\Gamma(n+2+\gamma)} {}_2F_1 \left[\begin{matrix} \theta, \theta \\ n+2+\gamma \end{matrix} ; |\eta|^2 \right]$$

holds for all $\eta \in \mathbb{B}_{n+1}$.

Proof. See [18, Corollary 2.4]. \square

Lemma 16. *Let $\gamma > -1$ and $\theta \in \mathbb{R}$. Then*

$$(3.13) \quad \int_{\mathcal{U}^n} \frac{\rho(w, w)^\gamma dV(w)}{|\rho(z, w)|^{2\theta} |\rho(w, \mathbf{i})|^{2(n+2+\gamma-\theta)}} \\ = \frac{4\pi^{n+1}\Gamma(1+\gamma)}{\Gamma(n+2+\gamma)} {}_2F_1 \left[\begin{matrix} \theta, \theta \\ n+2+\gamma \end{matrix} ; 1 - \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} \right] |\rho(z, \mathbf{i})|^{-2\theta}$$

holds for all $z \in \mathcal{U}^n$.

Proof. For fixed $z \in \mathcal{U}^n$, consider the integral

$$I(z) := \int_{\mathbb{B}^{n+1}} \frac{(1-|\xi|^2)^\gamma}{|1-\Psi(z) \cdot \xi|^{2\theta}} dV(\xi).$$

Making the change of variables $\xi = \Psi(w)$ and using (2.7)–(2.9), we get

$$(3.14) \quad I(z) = \int_{\mathcal{U}^n} \left(\frac{\rho(w, w)}{|\rho(w, \mathbf{i})|^2} \right)^\gamma \left| \frac{\rho(z, w)}{\rho(z, \mathbf{i})\rho(\mathbf{i}, w)} \right|^{-2\theta} \frac{1}{4|\rho(w, \mathbf{i})|^{2(n+2)}} dV(w) \\ = \frac{|\rho(z, \mathbf{i})|^{2\theta}}{4} \int_{\mathcal{U}^n} \frac{\rho(w, w)^\gamma dV(w)}{|\rho(z, w)|^{2\theta} |\rho(\mathbf{i}, w)|^{2(n+2+\gamma-\theta)}}.$$

On the other hand, by (3.12) and (2.8),

$$(3.15) \quad I(z) = \frac{\pi^{n+1}\Gamma(1+\gamma)}{\Gamma(n+2+\gamma)} {}_2F_1 \left[\begin{matrix} \theta, \theta \\ n+2+\gamma \end{matrix} ; |\Psi(z)|^2 \right] \\ = \frac{\pi^{n+1}\Gamma(1+\gamma)}{\Gamma(n+2+\gamma)} {}_2F_1 \left[\begin{matrix} \theta, \theta \\ n+2+\gamma \end{matrix} ; 1 - \frac{\rho(z, z)}{|\rho(z, \mathbf{i})|^2} \right].$$

The identity (3.13) now follows by comparing (3.14) and (3.15). \square

Lemma 17. *If $\kappa > -n-2$ and $\theta > \max\{\kappa, 0\}$, then*

$$(3.16) \quad \int_{\mathcal{U}^n} \frac{dV(w)}{\rho(z, w)^{n+2} \rho(\mathbf{i}, w)^\kappa \rho(w, \mathbf{i})^{\theta-\kappa}} \\ = \frac{4\pi^{n+1}\Gamma(\theta)}{\Gamma(\theta-\kappa)\Gamma(n+2+\kappa)} {}_2F_1 \left[\begin{matrix} \theta, \kappa \\ n+2+\kappa \end{matrix} ; 1 - \rho(z, \mathbf{i})^{-1} \right] \rho(z, \mathbf{i})^{-\theta}$$

holds for all $z \in \mathcal{U}^n$.

Proof. Making the change of variables $w = \Phi(\xi)$ and using (2.5) and (2.6), the integral becomes

$$\int_{\mathbb{B}_{n+1}} \left[\frac{1-\Psi(z) \cdot \bar{\xi}}{(1+[\Psi(z)]_{n+1})(1+\xi_{n+1})} \right]^{-n-2} (1+\xi_{n+1})^{\theta-\kappa} (1+\bar{\xi}_{n+1})^\kappa \frac{4V(\xi)}{|1+\xi_{n+1}|^{2(n+2)}} \\ = 4(1+[\Psi(z)]_{n+1})^{n+2} \int_{\mathbb{B}_{n+1}} \frac{dV(\xi)}{(1-\Psi(z) \cdot \bar{\xi})^{n+2} (1+\bar{\xi}_{n+1})^{-\kappa} (1+\xi_{n+1})^{n+2-\theta+\kappa}}.$$

By (3.11), this equals

$$\begin{aligned} & \frac{4\pi^{n+1}\Gamma(\theta)}{\Gamma(n+2+\kappa)\Gamma(\theta-\kappa)}(1+[\Psi(z)]_{n+1})^{n+2} {}_2F_1\left[\begin{matrix} n+2-\theta+\kappa, n+2 \\ n+2+\kappa \end{matrix}; -[\Psi(z)]_{n+1}\right] \\ &= \frac{4\pi^{n+1}\Gamma(\theta)}{\Gamma(n+2+\kappa)\Gamma(\theta-\kappa)}(1+[\Psi(z)]_{n+1})^\theta {}_2F_1\left[\begin{matrix} \theta, \kappa \\ n+2+\kappa \end{matrix}; -[\Psi(z)]_{n+1}\right], \end{aligned}$$

and (3.16) is proved, in view of that $1+[\Psi(z)]_{n+1} = \rho(z, \mathbf{i})^{-1}$. \square

4. PROOF OF THEOREM 2

4.1. The upper estimate. We first show that

$$\|T_\alpha\|_{p \rightarrow p} \leq \frac{(n+1)!\Gamma\left(1+\alpha-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}.$$

We shall distinguish two cases.

Case 1: $p = 1$. In this case, the assumption $p(1+\alpha) > 1$ implies that $\alpha > 0$. By Fubini's theorem and (3.9), we have

$$\begin{aligned} \|T_\alpha f\|_1 &= \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \left| \int_{\mathcal{U}^n} \frac{\rho(w, w)^\alpha}{|\rho(z, w)|^{n+2+\alpha}} f(w) dV(w) \right| dV(z) \\ &\leq \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} |f(w)| \rho(w, w)^\alpha \left\{ \int_{\mathcal{U}^n} \frac{dV(z)}{|\rho(z, w)|^{n+2+\alpha}} \right\} dV(w) \\ &= \frac{(n+1)!\Gamma(\alpha)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)} \|f\|_1. \end{aligned}$$

Case 2: $1 < p < \infty$. The proof appeals to Schur's test (Lemma 8). With

$$K(z, w) = \frac{(n+1)!}{4\pi^{n+1}} \frac{\rho(w, w)^\alpha}{|\rho(z, w)|^{n+2+\alpha}}$$

and

$$g(z) = \rho(z, z)^{-\frac{1}{pq}},$$

where $q = p/(p-1)$, using (3.9), we see that

$$\begin{aligned} \int_{\mathcal{U}^n} K(z, w) g(w)^q dV(w) &= \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \frac{\rho(w, w)^{\alpha-\frac{1}{p}}}{|\rho(z, w)|^{n+2+\alpha}} dV(w) \\ &= \frac{(n+1)!\Gamma\left(1+\alpha-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)} \rho(z, z)^{-\frac{1}{p}} \\ &= \frac{(n+1)!\Gamma\left(1+\alpha-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)} g(z)^q \end{aligned}$$

holds for every $z \in \mathcal{U}^n$. Similarly,

$$\begin{aligned} \int_{\mathcal{U}^n} K(z, w) g(z)^p dV(z) &= \frac{(n+1)!}{4\pi^{n+1}} \rho(w, w)^\alpha \int_{\mathcal{U}^n} \frac{\rho(z, z)^{-\frac{1}{q}}}{|\rho(z, w)|^{n+2+\alpha}} dV(z) \\ &= \frac{(n+1)! \Gamma\left(1 - \frac{1}{q}\right) \Gamma\left(\alpha + \frac{1}{q}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)} \rho(w, w)^{-\frac{1}{q}} \\ &= \frac{(n+1)! \Gamma\left(1 + \alpha - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)} g(w)^p \end{aligned}$$

holds for every $w \in \mathcal{U}^n$. Thus, by Schur's test, this yields the desired upper bound.

4.2. The lower estimate. We now proceed to show

$$\|T_\alpha\|_{p \rightarrow p} \geq \frac{(n+1)! \Gamma\left(1 + \alpha - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}.$$

Case I: $(n+2+\alpha)p > n+3$. For notational convenience, we write $\beta := \frac{n+2+\alpha}{2}$. For $0 < t < \frac{1}{p}$, we consider the function

$$\psi_t(w) := \frac{\rho(w, w)^{-t}}{|\rho(w, \mathbf{i})|^{2(\beta-t)}}, \quad w \in \mathcal{U}^n.$$

Note that the assumption $(n+2+\alpha)p > n+3$ implies that $(2\beta-t)p - n - 2 > 0$, which guarantees that $\psi_t \in L^p(\mathcal{U}^n)$ for all $t \in (0, \frac{1}{p})$. Indeed, by applying (3.9) with $\gamma = -tp$ and $\theta = (2\beta-t)p - n - 2$, we have

$$\|\psi_t\|_p^p = \frac{4\pi^{n+1} \Gamma(1-tp) \Gamma((2\beta-t)p - n - 2)}{\Gamma^2((\beta-t)p)}.$$

This implies that

$$(4.1) \quad \lim_{t \nearrow \frac{1}{p}} \|\psi_t\|_p^{-p} = 0,$$

since $\Gamma(1-tp) \rightarrow \infty$ as $t \nearrow \frac{1}{p}$.

Next, applying (3.13) with $\gamma = 2\beta - t - n - 2$ and $\theta = \beta$, we obtain

$$\begin{aligned} (T_\alpha \psi_t)(z) &= \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \frac{\rho(w, w)^{2\beta-t-n-2}}{|\rho(z, w)|^{2\beta} |\rho(w, \mathbf{i})|^{2(\beta-t)}} dV(w) \\ &= \frac{(n+1)! \Gamma(2\beta-t-n-1)}{\Gamma(2\beta-t)} {}_2F_1 \left[\begin{matrix} \beta, \beta \\ 2\beta-t \end{matrix} ; 1 - \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} \right] |\rho(\mathbf{i}, z)|^{-2\beta} \\ &= \frac{(n+1)! \Gamma(2\beta-t-n-1)}{\Gamma(2\beta-t)} \\ &\quad \times {}_2F_1 \left[\begin{matrix} \beta-t, \beta-t \\ 2\beta-t \end{matrix} ; 1 - \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} \right] \frac{\rho(z, z)^{-t}}{|\rho(z, \mathbf{i})|^{2(\beta-t)}}. \end{aligned}$$

The last equality follows from (2.16). For simplicity, we rewrite this as

$$(4.2) \quad (T_\alpha \psi_t)(z) = \frac{(n+1)! \Gamma(2\beta-t-n-1) \Gamma(t)}{\Gamma^2(\beta)} H \left(t, 1 - \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} \right) \psi_t(z),$$

with

$$H(t, \lambda) := \frac{\Gamma^2(\beta)}{\Gamma(2\beta - t)\Gamma(t)} {}_2F_1 \left[\begin{matrix} \beta - t, \beta - t \\ 2\beta - t \end{matrix} ; \lambda \right].$$

Note that the above hypergeometric function is increasing on the interval $[0, 1)$, since its Taylor coefficients are all positive.

Now we think of $H(t, \lambda)$ as a family of continuous functions of t on $[\frac{1}{2p}, \frac{1}{p}]$ indexed by $\lambda \in [0, 1)$. These functions tend monotonically to the constant function 1 pointwise as $\lambda \nearrow 1$, by (2.15). Moreover, the convergence is uniform in t on $[\frac{1}{2p}, \frac{1}{p}]$, by Dini's theorem.

It follows that for any $\epsilon > 0$, there exists a $\delta > 0$, sufficiently small and independent of $t \in [\frac{1}{2p}, \frac{1}{p}]$, such that

$$H(t, \lambda) \geq 1 - \epsilon, \quad \text{provided } \lambda > 1 - \delta.$$

This, together with (4.2), shows that

$$(4.3) \quad |(T_\alpha \psi_t)(z)| \geq (1 - \epsilon) \frac{(n+1)! \Gamma(2\beta - t - n - 1) \Gamma(t)}{\Gamma^2(\beta)} \chi_{E_\delta}(z) |\psi_t(z)|$$

holds for all $z \in \mathcal{U}^n$ and all $t \in [\frac{1}{2p}, \frac{1}{p}]$, where

$$(4.4) \quad E_\delta := \left\{ z \in \mathcal{U} : \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} < \delta \right\},$$

and χ_{E_δ} denotes the indicator function for the set E_δ . Consequently,

$$(4.5) \quad \|T_\alpha\|_{p \rightarrow p} \geq (1 - \epsilon) \frac{(n+1)! \Gamma(2\beta - t - n - 1) \Gamma(t)}{\Gamma^2(\beta)} \left\{ 1 - \|\psi_t\|_p^{-p} \int_{\mathcal{U}^n \setminus E_\delta} |\psi_t|^p dV \right\}^{\frac{1}{p}},$$

since $\|T_\alpha\|_{p \rightarrow p} \geq \|T_\alpha \psi_t\|_p / \|\psi_t\|_p$.

Making the change of variables $w = \Phi(\xi)$ and using (2.8) and (2.9), we have

$$\begin{aligned} \int_{\mathcal{U}^n \setminus E_\delta} |\psi_t|^p dV &= \int_{\mathcal{U}^n \setminus E_\delta} \frac{\rho(w, w)^{-tp}}{|\rho(\mathbf{i}, w)|^{2(\beta-t)p}} dV(w) \\ &= \int_{|\xi| \leq \sqrt{1-\delta}} \left(\frac{1 - |\xi|^2}{|1 + \xi_{n+1}|^2} \right)^{-tp} |1 + \xi_{n+1}|^{2(\beta-t)p} \frac{4dV(\xi)}{|1 + \xi_{n+1}|^{2(n+2)}} \\ &= 4 \int_{|\xi| \leq \sqrt{1-\delta}} \frac{(1 - |\xi|^2)^{-tp}}{|1 + \xi_{n+1}|^{2(n+2-\beta p)}} dV(\xi). \end{aligned}$$

It is easily seen that if $|\xi| \leq \sqrt{1-\delta}$ then

$$|1 + \xi_{n+1}|^{2(\beta p - n - 2)} \leq \max \left\{ \left(\frac{2}{\delta} \right)^{2(n+2-\beta p)}, 2^{2(\beta p - n - 2)} \right\}$$

and

$$(1 - |\xi|^2)^{-tp} \leq \delta^{-tp} \leq \delta^{-1},$$

hold for all $t \in [\frac{1}{2p}, \frac{1}{p})$. It follows that

$$\sup_{t \in [\frac{1}{2p}, \frac{1}{p})} \int_{\mathcal{U}^n \setminus E_\delta} |\psi_t|^p dV \leq C_\delta,$$

with

$$C_\delta := \frac{2\pi^{n+1}}{(n+1)!} \frac{(1-\delta)^{n+1}}{\delta} \max \left\{ \left(\frac{2}{\delta} \right)^{2(n+2-\beta p)}, 2^{2(\beta p - n - 2)} \right\}.$$

Keep in mind that δ is independent of $t \in [\frac{1}{2p}, \frac{1}{p})$. Combining this with (4.6), we obtain

$$\lim_{t \nearrow \frac{1}{p}} \|\psi_t\|_p^{-p} \int_{\mathcal{U}^n \setminus E_\delta} |\psi_t|^p dV = 0.$$

Now, letting $t \nearrow \frac{1}{p}$ in (4.5), we conclude that

$$\begin{aligned} \|T_\alpha\|_{p \rightarrow p} &\geq (1-\epsilon) \frac{(n+1)! \Gamma\left(2\beta - n - 1 - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2(\beta)} \\ &= (1-\epsilon) \frac{(n+1)! \Gamma\left(1 + \alpha - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}. \end{aligned}$$

Since ϵ is arbitrary, this yields

$$\|T_\alpha\|_{p \rightarrow p} \geq \frac{(n+1)! \Gamma\left(1 + \alpha - \frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}.$$

Case II: $(n+2+\alpha)p \leq n+3$. In this case, the above test function ψ_t does not belong to $L^p(\mathcal{U}^n)$. Instead, for $t \in \left(\max\{0, -\alpha\}, \frac{1}{q}\right)$, we consider the function

$$\tilde{\psi}_t(w) := \frac{\rho(w, w)^{-t}}{|\rho(w, \mathbf{i})|^{2(\tilde{\beta}-t)}}, \quad w \in \mathcal{U}^n,$$

where

$$q := \frac{p}{p-1} \quad \text{and} \quad \tilde{\beta} := \frac{n+2-\alpha}{2}.$$

Note that the assumption $(n+2+\alpha)p \leq n+3$ implies $(2\tilde{\beta}-t)q - n - 2 > 0$, which guarantees that $\tilde{\psi}_t \in L^q(\mathcal{U}^n)$ for all $t \in (0, \frac{1}{q})$. Indeed, by applying (3.9) with $\gamma = -tq$ and $\theta = (2\tilde{\beta}-t)q - n - 2$, we have

$$\|\tilde{\psi}_t\|_q^q = \frac{4\pi^{n+1} \Gamma(1-tq) \Gamma((2\tilde{\beta}-t)q - n - 2)}{\Gamma^2((\tilde{\beta}-t)q)}.$$

This also implies that

$$(4.6) \quad \lim_{t \nearrow \frac{1}{q}} \|\tilde{\psi}_t\|_q^{-q} = 0,$$

since $\Gamma(1-tq) \rightarrow \infty$ as $t \nearrow \frac{1}{q}$.

Note that the adjoint of T_α is given by

$$(T_\alpha^* f)(z) = \frac{(n+1)!}{4\pi^{n+1}} \rho(z, z)^\alpha \int_{\mathcal{U}^n} \frac{f(w)}{|\rho(z, w)|^{n+2+\alpha}} dV(w).$$

Applying (3.13) with $\gamma = -t$ and $\theta = n + 2 - \tilde{\beta}$, we obtain

$$\begin{aligned} (T_\alpha^* \tilde{\psi}_t)(z) &= \frac{(n+1)!}{4\pi^{n+1}} \rho(z, z)^{n+2-2\tilde{\beta}} \int_{\mathcal{U}^n} \frac{\rho(w, w)^{-t}}{|\rho(z, w)|^{2(n+2-\tilde{\beta})} |\rho(w, \mathbf{i})|^{2(\tilde{\beta}-t)}} dV(w) \\ &= \frac{(n+1)! \Gamma(1-t)}{\Gamma(n+2-t)} {}_2F_1 \left[\begin{matrix} \tilde{\beta}-t, \tilde{\beta}-t \\ n+2-t \end{matrix} ; 1 - \frac{\rho(z, z)}{|\rho(\mathbf{i}, z)|^2} \right] \frac{\rho(z, z)^{-t}}{|\rho(z, \mathbf{i})|^{2(\tilde{\beta}-t)}}. \end{aligned}$$

A similar argument as in Case I shows that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|(T_\alpha^* \tilde{\psi}_t)(z)| \geq (1-\epsilon) \frac{(n+1)! \Gamma(n+2+t-2\tilde{\beta}) \Gamma(1-t)}{\Gamma^2(n+2-\tilde{\beta})} \chi_{E_\delta}(z) |\tilde{\psi}_t(z)|$$

holds for all $z \in \mathcal{U}^n$ and all $t \in \left[\max\{-\alpha, \frac{1}{2q}\}, \frac{1}{q} \right)$, where E_δ is as in (4.4) and χ_{E_δ} denotes the indicator function for the set E_δ . Consequently,

$$\begin{aligned} \|T_\alpha^*\|_{q \rightarrow q} &\geq (1-\epsilon) \frac{(n+1)! \Gamma(n+2-2\tilde{\beta}+t) \Gamma(1-t)}{\Gamma^2(n+2-\tilde{\beta})} \\ &\quad \times \left\{ 1 - \|\tilde{\psi}_t\|_q^{-q} \int_{\mathcal{U}^n \setminus E_\delta} |\tilde{\psi}_t|^q dV \right\}^{\frac{1}{q}}. \end{aligned} \quad (4.7)$$

Exactly as in Case I, we can let $t \nearrow \frac{1}{q}$ in (4.7) to yield

$$\begin{aligned} \|T_\alpha^*\|_{q \rightarrow q} &\geq (1-\epsilon) \frac{(n+1)! \Gamma\left(n+2-2\tilde{\beta}+\frac{1}{q}\right) \Gamma\left(1-\frac{1}{q}\right)}{\Gamma^2(n+2-\tilde{\beta})} \\ &= (1-\epsilon) \frac{(n+1)! \Gamma\left(1+\alpha-\frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}. \end{aligned}$$

Since ϵ is arbitrary, this yields

$$\|T_\alpha\|_{p \rightarrow p} = \|T_\alpha^*\|_{q \rightarrow q} \geq \frac{(n+1)! \Gamma\left(1+\alpha-\frac{1}{p}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma^2\left(\frac{n+2+\alpha}{2}\right)}.$$

The proof is now complete.

5. PROOF OF THEOREM 1

As is mentioned in the introduction, the second inequality in (1.2) follows immediately from Theorem 2, so we prove only the first inequality.

Further, we only need to consider the case when $p \geq 2$, and the case when $1 < p < 2$ then follows from the duality.

For notational convenience, we put

$$\kappa := (n+2) \left(\frac{1}{2} - \frac{1}{p} \right) \quad \text{and} \quad \theta := \frac{n+2}{p}.$$

Note that $\theta + \kappa = \frac{n+2}{2}$.

For $0 < \epsilon < \kappa$, we consider the function

$$f_\epsilon(z) := \frac{1}{\rho(\mathbf{i}, z)^{\epsilon-\kappa} \rho(z, \mathbf{i})^{\theta+\kappa}}, \quad z \in \mathcal{U}^n.$$

Using (3.16) we get

$$\begin{aligned} (P_{\mathcal{U}^n} f_\epsilon)(z) &= \frac{(n+1)!}{4\pi^{n+1}} \int_{\mathcal{U}^n} \frac{dV(w)}{\rho(z, w)^{n+2} \rho(\mathbf{i}, w)^{\epsilon-\kappa} \rho(w, \mathbf{i})^{\theta+\kappa}} \\ &= \frac{(n+1)! \Gamma(\theta + \epsilon)}{\Gamma(\theta + \kappa) \Gamma(n+2 + \epsilon - \kappa)} {}_2F_1 \left[\begin{matrix} \epsilon - \kappa, \theta + \epsilon \\ n+2 + \epsilon - \kappa \end{matrix} ; 1 - \frac{1}{\rho(z, \mathbf{i})} \right] \frac{1}{\rho(z, \mathbf{i})^{\theta+\epsilon}}. \end{aligned}$$

Now we consider the functions

$$\begin{aligned} g_\epsilon(z) &:= \frac{(n+1)! \Gamma(\theta + \epsilon)}{\Gamma(\theta + \kappa) \Gamma(n+2 + \epsilon - \kappa)} {}_2F_1 \left[\begin{matrix} \epsilon - \kappa, \theta + \epsilon \\ n+2 + \epsilon - \kappa \end{matrix} ; 1 \right] \frac{1}{\rho(z, \mathbf{i})^{\theta+\epsilon}}, \\ &= \frac{\Gamma(\theta + \epsilon) \Gamma(n+2 - \theta - \epsilon)}{\Gamma(\theta + \kappa) \Gamma(n+2 - \kappa - \theta)} \frac{1}{\rho(z, \mathbf{i})^{\theta+\epsilon}}, \end{aligned}$$

and

$$\begin{aligned} h_\epsilon(z) &:= \frac{(n+1)! \Gamma(\theta + \epsilon)}{\Gamma(\theta + \kappa) \Gamma(n+2 + \epsilon - \kappa)} \frac{1}{\rho(z, \mathbf{i})^{\theta+\epsilon}}, \\ &\quad \times \left\{ {}_2F_1 \left[\begin{matrix} \epsilon - \kappa, \theta + \epsilon \\ n+2 + \epsilon - \kappa \end{matrix} ; 1 - \frac{1}{\rho(z, \mathbf{i})} \right] - {}_2F_1 \left[\begin{matrix} \epsilon - \kappa, \theta + \epsilon \\ n+2 + \epsilon - \kappa \end{matrix} ; 1 \right] \right\}. \end{aligned}$$

It is clear that $P_{\mathcal{U}^n} f_\epsilon = g_\epsilon + h_\epsilon$, and hence

$$\|P_{\mathcal{U}^n}\|_p \geq \limsup_{\epsilon \rightarrow 0^+} \frac{\|P_{\mathcal{U}^n} f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \limsup_{\epsilon \rightarrow 0^+} \left(\frac{\|g_\epsilon\|_p}{\|f_\epsilon\|_p} - \frac{\|h_\epsilon\|_p}{\|f_\epsilon\|_p} \right).$$

It is clear that

$$\|g_\epsilon\|_p = \frac{\Gamma(\theta + \epsilon) \Gamma(n+2 - \theta - \epsilon)}{\Gamma(\theta + \kappa) \Gamma(n+2 - \kappa - \theta)} \|f_\epsilon\|_p,$$

and hence

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{\|g_\epsilon\|_p}{\|f_\epsilon\|_p} &= \frac{\Gamma(\theta) \Gamma(n+2 - \theta)}{\Gamma(\theta + \kappa) \Gamma(n+2 - \kappa - \theta)} \\ &= \frac{\Gamma\left(\frac{n+2}{p}\right) \Gamma\left(n+2 - \frac{n+2}{p}\right)}{\Gamma^2\left(\frac{n+2}{2}\right)}. \end{aligned}$$

Thus we are reduced to proving

$$\limsup_{\epsilon \rightarrow 0^+} \frac{\|h_\epsilon\|_p}{\|f_\epsilon\|_p} = 0.$$

Using (3.9), we see that

$$\|f_\epsilon\|_p^p = \int_{\mathcal{U}^n} \frac{dV(w)}{|\rho(\mathbf{i}, w)|^{n+2+p\epsilon}} = \frac{4\pi^{n+1} \Gamma(p\epsilon)}{\Gamma^2\left(\frac{n+2+p\epsilon}{2}\right)} \rightarrow \infty \quad \text{as } \epsilon \rightarrow 0^+.$$

Hence, the proof is completed by showing that

$$(5.1) \quad \sup_{0 < \epsilon < \kappa} \|h_\epsilon\|_p < \infty.$$

Indeed, by Lemma 9, it is easy to check that

$$|h_\epsilon(z)| \leq \frac{C(\epsilon)}{|\rho(z, \mathbf{i})|^{\theta+1+\epsilon}}$$

holds for all $z \in \mathcal{U}^n$, with the constant

$$C(\epsilon) := \frac{|\epsilon - \kappa| \Gamma(\theta + \epsilon)}{\Gamma(\theta + \kappa)} \max \left\{ \frac{2^{\kappa - \epsilon - 1} (n+1)!}{\Gamma(n+2+\epsilon-\kappa)}, \frac{(n+1) \Gamma(n+1-\theta-\epsilon)}{\Gamma(n+2-\kappa-\theta)} \right\}.$$

Therefore, by (3.9), we obtain

$$\|h_\epsilon\|_p^p \leq C(\epsilon)^p \int_{\mathcal{U}^n} \frac{dV(w)}{|\rho(w, \mathbf{i})|^{n+2+p(1+\epsilon)}} = C(\epsilon)^p \cdot \frac{4\pi^{n+1} \Gamma(p(1+\epsilon))}{\Gamma^2\left(\frac{n+2+p(1+\epsilon)}{2}\right)},$$

and (5.1) easily follows. The proof is now complete.

6. PROOF OF THEOREM 5

We first present two auxiliary lemmas, which are interesting for their own sake.

Lemma 18. *If F is plurisubharmonic in \mathcal{U}^n ($n \geq 1$) and $F \geq 0$, then*

$$(6.1) \quad \pi \int_{\mathcal{U}^{n-1}} F(0, w) dm_{2n}(w) \leq \sup_{t > 0} \int_{b\mathcal{U}^n} F(z + t\mathbf{i}) d\beta(z).$$

Proof. Note that $(\sqrt{\rho_{n-1}(w)} e^{i\theta}, w) \in b\mathcal{U}^n$ for any $w \in \mathcal{U}^{n-1}$ and any $\theta \in [0, 2\pi)$. For $t > 0$,

$$F((0, w) + t\mathbf{i}) \leq \frac{1}{2\pi} \int_0^{2\pi} F((\sqrt{\rho_{n-1}(w)} e^{i\theta}, w) + t\mathbf{i}) d\theta,$$

since f is plurisubharmonic. By Lemma (3.7), it follows that

$$(6.2) \quad \begin{aligned} \pi \int_{\mathcal{U}^{n-1}} F((0, w) + t\mathbf{i}) dm_{2n}(w) \\ \leq \pi \int_{\mathcal{U}^{n-1}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} F((\sqrt{\rho_{n-1}(w)} e^{i\theta}, w) + t\mathbf{i}) d\theta \right\} dm_{2n}(w) \\ = \int_{b\mathcal{U}^n} F(z + t\mathbf{i}) d\beta(z). \end{aligned}$$

Note that

$$\int_{\mathcal{U}^{n-1}} F((0, w) + t\mathbf{i}) dm_{2n}(w) = \int_{\{\rho_{n-1}(w) > t\}} F(0, w) dm_{2n}(w).$$

Hence (6.1) follows from (6.2) as $t \searrow 0$. \square

Let f and g be functions with domains \mathcal{U}^n and \mathcal{U}^{n-1} , respectively. We define a restriction operator \mathfrak{R} and an extension operator \mathfrak{E} by

$$\begin{aligned} (\mathfrak{R}f)(\tilde{z}) &= f(0, \tilde{z}) & (\tilde{z} \in \mathcal{U}^{n-1}), \\ (\mathfrak{E}g)(z_1, \tilde{z}) &= g(\tilde{z}) & (z = (z_1, \tilde{z}) \in \mathcal{U}^n). \end{aligned}$$

Lemma 19. *Suppose $n \geq 1$ and $0 < p < \infty$. Then the restriction operator \mathfrak{R} maps $H^p(\mathcal{U}^n)$ boundedly onto $A^p(\mathcal{U}^{n-1})$, with operator norm equal to $\pi^{-\frac{1}{p}}$.*

Proof. Let $f \in H^p(\mathcal{U}^n)$. Applying Proposition 18 to $F = |f|^p$, we obtain

$$\|\Re f\|_{A^p(\mathcal{U}^{n-1})}^p \leq \pi^{-1} \|f\|_{H^p(\mathcal{U}^n)}^p.$$

This means that \Re maps $H^p(\mathcal{U}^n)$ boundedly into $A^p(\mathcal{U}^{n-1})$, with operator norm at most $\pi^{-\frac{1}{p}}$.

To see that \Re maps $H^p(\mathcal{U}^n)$ onto $A^p(\mathcal{U}^{n-1})$, let $f \in A^p(\mathcal{U}^{n-1})$. Since $f = \Re \mathfrak{E}f$, it suffices for us to show that $\mathfrak{E}f$ belongs to $H^p(\mathcal{U}^n)$. For any $t > 0$, again by the identity (3.7), we have

$$\begin{aligned} & \int_{b\mathcal{U}^n} |\mathfrak{E}f(z + t\mathbf{i})|^p d\beta(z) \\ &= \pi \int_{\mathcal{U}^{n-1}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \mathfrak{E}f((\sqrt{\rho_{n-1}(w)} e^{i\theta}, w) + t\mathbf{i}) \right|^p d\theta \right\} dm_{2n}(w) \\ &= \pi \int_{\mathcal{U}^{n-1}} |f(w + t\tilde{\mathbf{i}})|^p dm_{2n}(w) \quad (\tilde{\mathbf{i}} = (\underbrace{0, \dots, 0}_{n-1}, i)) \\ &= \pi \int_{\{\rho_{n-1}(w) > t\}} |f(w)|^p dm_{2n}(w) \leq \pi \|f\|_{A^p(\mathcal{U}^{n-1})}^p, \end{aligned}$$

which completes the proof. \square

We now turn to the proof of Theorem 5.

Proof of Theorem 5. Let $f \in L^p(\mathcal{U}^{n-1})$ be arbitrary, with $\|f\|_{L^p(\mathcal{U}^{n-1})} = 1$. Put

$$g := \pi^{-\frac{1}{p}} \mathfrak{E}f$$

By (3.3), we see that $g \in L^p(b\mathcal{U}^n)$ and

$$\|g\|_{L^p(b\mathcal{U}^n)} = \|f\|_{L^p(\mathcal{U}^{n-1})} = 1.$$

Note also that

$$S_{\mathcal{U}^n}((0, \tilde{z}), u) g(u) = \pi^{-1-\frac{1}{p}} K_{\mathcal{U}^{n-1}}(\tilde{z}, \tilde{u}) f(\tilde{u})$$

holds for any $\tilde{z} \in \mathcal{U}^{n-1}$ and $u = (u_1, \tilde{u}) \in b\mathcal{U}^n$, where $K_{\mathcal{U}^{n-1}}(\cdot, \cdot)$ is the Bergman kernel for \mathcal{U}^{n-1} and $S_{\mathcal{U}^n}(\cdot, \cdot)$ is the Cauchy-Szegö kernel for \mathcal{U}^n . It follows that

$$\begin{aligned} (C_{\mathcal{U}^n} g)((0, \tilde{z})) &= \pi^{-1-\frac{1}{p}} \int_{b\mathcal{U}^n} K_{\mathcal{U}^{n-1}}(\tilde{z}, \tilde{u}) f(\tilde{u}) d\beta(u) \\ &= \pi^{-\frac{1}{p}} \int_{\mathcal{U}^{n-1}} K_{\mathcal{U}^{n-1}}(\tilde{z}, w) f(w) dm_{2n}(w) \\ &= \pi^{-\frac{1}{p}} (P_{\mathcal{U}^{n-1}} f)(\tilde{z}), \end{aligned}$$

where in the second equality we used (3.3). Thus, by Theorem 19,

$$\|P_{\mathcal{U}^{n-1}} f\|_{A^p(\mathcal{U}^{n-1})} = \pi \|\Re(C_{\mathcal{U}^n} g)\|_{A^p(\mathcal{U}^{n-1})} \leq \|C_{\mathcal{U}^n} g\|_{H^p(\mathcal{U}^n)}.$$

Taking supremum over all $f \in L^p(\mathcal{U}^{n-1})$ with $\|f\|_{L^p(\mathcal{U}^{n-1})} = 1$ yields (1.6). \square

REFERENCES

- [1] G.E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
- [2] R. R. Coifman and R. Rochberg, Representation theorem for holomorphic and harmonic functions in L^p , pp. 12-66, *Astérisque*, 77, Soc. Math. France, Paris, 1980.
- [3] M. Dostanić, Norm of Berezin transform on L^p space, *J. Anal. Math.*, 104 (2008), 13–23.
- [4] M. Dostanić, Two sided norm estimate of the Bergman projection on L^p -spaces, *Czechoslovak Math. J.* 58(133)(2008), 569–575.
- [5] M. Dostanić, Integral operators induced by Bergman type kernels in the half plane. *Asymptot. Anal.* 67 (2010), no.3-4, 217–228.
- [6] F. Forelli, Measures whose Poisson integrals are pluriharmonic, *Illinois J. Math.* 18, 373–388 (1974).
- [7] S. G. Gindikin, Analysis in homogeneous domains, *Russian Math. Surveys* 19(4) (1964), 1–89.
- [8] B. Hollenbeck and I. E. Verbitsky, Best constants for the Riesz projection, *Journal of Functional Analysis* 175 (2000), 370–392.
- [9] D. Kalaj, M. Marković, Norm of the Bergman projection, *Math. Scand.* 115(2014), 143–160.
- [10] D. Kalaj, D. Vujadinović, Norm of the Bergman projection onto the Bloch space, *J. Operator Theory* 73 (2015), no. 1, 113–126.
- [11] A. Korányi, The Poisson integral for generalized half-planes and bounded symmetric domains, *Ann. of Math.*, 82(2) (1965), 332–350.
- [12] A. Korányi and E. M. Stein, Fatou’s theorem for generalized half-planes, *Ann. Scuola Norm. Sup. Pisa* 22 (1968), 107–112.
- [13] A. Korányi and S. Vági, Singular Integrals on Homogeneous Spaces and Some Problems of Classical Analysis, *Ann. Scuola Norm. Sup. Pisa*, 25(4) (1971), 575–648.
- [14] A. Korányi and J. A. Wolf, Realization of hermitian symmetric spaces as generalized half-planes, *Ann. of Math.* 81 (1965), 265–288.
- [15] S. G. Krantz, Function theory of several complex variables. Reprint of the 1992 edition. AMS Chelsea Publishing, Providence, RI, 2001.
- [16] S. G. Krantz, Explorations in harmonic analysis: with applications to complex function theory and the Heisenberg group. Birkhäuser Boston, Inc., Boston, MA, 2009.
- [17] N. Krupnik, Survey on the best constants in the theory of one-dimensional singular integral operators, *Operator Theory: Advances and Applications*, Vol. 202, 365–393, Birkhäuser, Springer Basel AG, 2010.
- [18] C. Liu, Sharp Forelli-Rudin estimates and the norm of the Bergman projection, *Journal of Functional Analysis*, 268 (2015), 255–277.
- [19] C. Liu, Norm of the Cauchy Transform, *Integral Equations Operator Theory*, 85 (2016), no. 3, 303–306.
- [20] C. Liu and L. Zhou, On the p -norm of the Berezin transform, *Illinois Journal of Mathematics*, 56(2) (2012), 497–505.
- [21] C. Liu and L. Zhou, On the p -norm of an integral operator in the half plane, *Canad. Math. Bull.*, 56(2013), 593–601.
- [22] A. Perälä, On the optimal constant for the Bergman projection onto the Bloch space, *Ann. Acad. Sci. Fenn. Math.* 37(2012), 245–249.
- [23] A. Perälä, Bloch spaces and the norm of the Bergman projection, *Ann. Acad. Sci. Fenn. Math.* 38(2013), 849–853.
- [24] A. Perälä, Sharp constant for the Bergman projection onto the minimal Möbius invariant space, *Arch. Math. (Basel)*. 102, (2014), no. 3, 263–270.
- [25] D. H. Phong and E. M. Stein, Hilbert integrals, singular integrals, and Radon transforms. II. *Invent. Math.* 86 (1986), no. 1, 75–113.
- [26] W. Rudin, Function Theory in the Unit Ball of \mathbb{C}^n , Reprint of the 1980 Edition, Springer, Berlin 2008.
- [27] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, Princeton, NJ, 1993.
- [28] D. Vujadinović, Some estimates for the norm of the Bergman projection on Besov spaces, *Integral Equations Operator Theory* 76(2013), 213–224.
- [29] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in Math., vol. 226, Springer, New York 2005.

- [30] K. Zhu, A sharp norm estimate of the Bergman projection on L^p -spaces, *Contemp. Math.* 404(2006), 199–205.
 - [31] K. Zhu, *Operator theory in function spaces*. Second edition. *Mathematical Surveys and Monographs*, 138. American Mathematical Society, Providence, RI, 2007.
- E-mail address:* `cwliu@ustc.edu.cn`

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA,,
HEFEI, ANHUI 230026, PEOPLE'S REPUBLIC OF CHINA, AND, WU WEN-TSUN KEY LABORATORY
OF MATHEMATICS, USTC, CHINESE ACADEMY OF SCIENCES